Pricing, resource allocation and quality of service in multi-class networks with competitive market model

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Abstract: This study considers a multi-class network resource in a competitive market where each user endowed with an initial budget will purchase bandwidth from each class of the network resource to maximise its utility function. The authors use revenue as the utility function for the service provider, and enhance Kelly’s utility function by including delay as the quality of service (QoS) parameter for users. A competitive equilibrium is reached when the total bandwidth is allocated, each user spends all its budget and the utility functions are independently and simultaneously maximised. The authors prove that such equilibriums always exist and, under fixed bandwidth supply for each class of service, the equilibrium is also unique. Furthermore, the authors discuss how to adjust the initial endowment of each user to meet his or her individual bandwidth constraint, either from constraint on the access network or the limitation of the user equipment. Under this bandwidth constraint condition, the proposed competitive equilibrium yields the price for each class of service, budget redistribution and bandwidth allocation among all users. The competitive market model presented provides a solution for pricing a multi-class network resource and allocating the resource that achieves both higher social utility and better individual satisfaction whereas maintaining the QoS of each class.

1 Introduction

Class-based network architectures like DiffServ [1] have become the most viable solution for providing quality-of-service (QoS) over IP networks. They have been proposed as the transport QoS architecture in IMS standardisation by 3GPP [2] and NGN release 1 standardisation by ETSI TISPAN [3]. In multi-class networks, there is clearly the need for incentives to be offered to users in order to encourage them to choose the amount of bandwidth from each class that is most appropriate to their needs, thereby discouraging the over-allocation, maximising the social welfare and individual utility. This objective is commonly achieved through pricing. In multi-class networks, pricing transcends its traditional role in cost recovery [4, 5] to become an important network management tool.

Using pricing as a tool to manage and study the economic viability of the network has been intensively investigated.

Studies in [6–13] use a centralised optimisation process to maximise the total user utility. In order to solve the scalability associated with this maximisation problem, Kelly et al. [6] form a distributed flow control algorithm using gradient ascent algorithm from optimisation theory which continuously informs the selfish users prices according to the network condition. Selfish users, who seek to maximise their own net benefit, are given the prices as right incentives to globally optimise the social benefits. However, these schemes assume the network provides only one type of service – best effort service – and an aggregate optimal resource allocation may not simultaneously optimise each user’s individual utility.

Auction-based mechanisms have been studied in [14, 15]. The smart market model is studied in [14], in which a packet is admitted if the bid exceeds the current marginal congestion cost. However, this mechanism only provides a priority relative to others and it does not promise QoS. The
The study of competitive economy equilibrium was first started by Walras [28] over a hundred years ago. In this problem, each user participating in the market initially has an endowment of some amount of each of \( n \) goods \( \mathbf{w} = (w_1, \ldots, w_n) \). Every agent sells the entire initial endowment and buys a bundle of goods \( \mathbf{x} = (x_1, \ldots, x_n) \) to maximise his utility function \( u_i(x_i) \). Subject to the following constraint, \( \mathbf{x} p^T \leq \mathbf{w} p^T \), Arrow and Debreu [29] show that there exists equilibrium prices \( \mathbf{p} = (p_1, \ldots, p_n) \) for the \( n \) goods such that the market is cleared (the demand of each of the goods equals the supply) if the utility function of each agent \( u_i \) were concave. Reference [30] provides an algorithm to compute this competitive economy equilibrium.

In modern networks, terminals/users no longer use one type of service. Instead, they consume multiple services at the same time. For example, when a user is playing an online game, he/she may also need to chat with his player or use text messaging. It is natural for us to model a multi-class network as a competitive market where agents are divided into two sets: users and the service provider. The service provider supports \( n \) types of services and each service has a different QoS guarantee. Users spend money to buy a bundle of \( n \) types of services and maximise their individual utility. An equilibrium is a set of prices for \( n \) services so that the market is cleared and each user’s utility is maximised. This model is a special case of Walras’ model when money is also considered as a good.

For general concave and homogeneous utility functions, the equilibrium problem is reduced to a social utility maximisation problem over a convex set defined by the supply–demand linear constraints and the equilibrium prices derived from the Lagrangian multipliers of these constraints [31–33]. References [34, 35] investigate the competitive market equilibrium with non-homogeneous utility functions which include goods purchased by other users. It means that each user’s utility function not only depends on his purchase action but also related to other users’ purchase choices. This is also the case for communication networks in that the performance obtained by any given network user is determined by all users’ traffic and service choices.

In this paper, we consider pricing, resource allocation and QoS provisioning in a multi-class network under the competitive economy model. We use revenue as the utility function for the service provider and enhance Kelly’s utility function by including a QoS parameter for users. Given the initial endowment for each user, we show that a competitive equilibrium (price for each class of service and bandwidth allocation among all users) for the competitive multi-class network resource market always exists. And under this equilibrium, both individual optimality and social economic efficiency is achieved in a way that all users’ utilities are simultaneously maximised. In other words, this equilibrium is Pareto optimal. We further show that under the fixed supply condition for each class of service, this equilibrium is unique which can be computed in polynomial time.

In addition, this paper also discusses how to adjust the initial endowment for each user to meet his/her bandwidth constraint (either from constraint on the access network or limitation of the user equipment). Under this constraint, the competitive equilibrium is the price for each class of service, budget redistribution and bandwidth allocation among all users. The equilibrium conditions are analysed and the existence of the equilibrium is also proved. A procedure to recompute the equilibrium is proposed.

The rest of the paper is organised as follows. Section 2 lists the mathematical notations used in this paper. In Section 3, we introduce our model of competitive multi-class network resource market. In this section, both the service provider’s and users’ utility functions are presented and investigated. Section 4 proves the existence and uniqueness of the competitive equilibrium of this market. In Section 5, we
discuss how to adjust the users’ initial budget under each user’s bandwidth constraint. Section 6 gives a numerical example resulting in the competitive equilibrium in pricing and resource allocation in multi-class networks, and Section 7 captures the conclusions of this paper.

2 Mathematical notations

Mathematical notations used in this paper are described in this section. We use \( \mathbb{R}^n \) to denote the \( n \)-dimensional Euclidean space and \( \mathbb{R}^+_n \) to denote the subset of \( \mathbb{R}^n \), where each coordinate is non-negative. We also use \( \mathbb{R} \) and \( \mathbb{R}_+ \) to represent the set of real numbers and the set of non-negative real numbers. Throughout this paper, for a vector \( x, x > 0 \) means that every component of \( x \) is larger than 0. Other comparative notations \(<, \geq, \leq \) are extended in a similar manner.

In this paper, we assume the network provides \( n \) services. Each class of service has a different QoS guarantee and is suitable for a different application. There are \( m \) users sharing the network resource. We use \( X \in \mathbb{R}^{m \times n} \) to denote the set of ordered \( m \)-tuples \( X = (x_1, \ldots, x_m) \) and use \( X^i \in \mathbb{R}^{(m-1) \times n} \) to denote the set of ordered \((m-1)\)-tuples \( X^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \), where \( x_i = (x_{i1}, \ldots, x_{im}) \in X^i \in \mathbb{R}^n \) for \( i = 1, \ldots, m \). For each user \( i \), we use \( u_i(x_i, \bar{x}_i) \) to denote his utility function, where \( x_i \in X_i \) and \( \bar{x}_i \in X_i \). It means that user \( i \)'s utility depends on his own action \( x_i \), as well as actions made by all other users \( \bar{x}_i \).

The following definitions [36] will be used in the proof of the existence and uniqueness of competitive equilibrium.

**Definition 1:** For a twice-differentiable function \( u_i: \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \) if for \( \nabla_x u_i \) is positive and \( \nabla^2_x u_i \) is negative, we have \( u_i \) as a monotonically increasing function and concave with respect to \( x_i \).

**Definition 2:** A function \( u_i: \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \) is concave if for any \( x, y \in \mathbb{R}^n_+ \) and any \( 0 \leq a \leq 1 \), we have \( u_i(ax + (1 - a)y) \geq au_i(x) + (1 - a)u_i(y) \). And if it is strictly concave, \( u_i(ax + (1 - a)y) > au_i(x) + (1 - a)u_i(y) \) for \( 0 < a < 1 \).

3 Competitive multi-class network resource market

In the competitive multi-class network resource market, as stated in Section 2, the network supports \( n \) types of services and there are \( m \) users in the network. There are three types of entities in the market: users, service provider and market.

Each user \( i \) is endowed a monetary budget \( \omega_i \ (> 0) \) and uses it to purchase some amount of each of \( n \) services \( x_i = (x_{i1}, \ldots, x_{in}) \) from an open market so as to maximise its own utility \( u_i(x_i, \bar{x}_i) \), where \( \bar{x}_i \) represents the amount of services obtained by all other users. The budget \( \omega_i \) for each user can represent the budgets for users to pay for the service as in [21]. As noted in Section 2, in this paper \( x_i \) is used to denote the amount of service \( j \) that user \( i \) purchases; \( x_i = (x_{i1}, \ldots, x_{in}) \) is an \( n \)-dimensional vector that shows some amount of each of \( n \) services obtained by user \( i \); \( x_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \) is an \((m-1) \times n\) vector and represents the amount of each of \( n \) services obtained by all other users except user \( i \).

We now consider the network service provider. It sets up the network and allocates limited network resource to \( n \) types of services from a convex and compact set \( C \) to maximise its utility.

The third entity, the market, sets unit prices \( \rho = (p_1, \ldots, p_n) \) for \( n \) types of services. Each type of service has a different QoS guarantee and \( p_i \) can be interpreted as the unit price for service \( i \). For example, \( p_1 = 1 \) and \( p_2 = 2 \) simply means that users can use one unit service 2 to trade for two units of service 1.

Then, user \( i \)'s \((i = 1, \ldots, m)\) individual utility maximisation problem is as follows

\[
\text{maximise} \ u_i(x_i, \bar{x}_i) \quad (1)
\]

subject to the following constraints

\[
\begin{align*}
x_i \rho^T \leq & \omega_i \\
x_i \geq & 0
\end{align*}
\]

Equation (2) shows that the total payment of the purchased \( n \) types of services should not exceed his endowed budget \( \omega_i \).

There are several assumptions about the utility functions used in the multi-class network as follows.

**Assumption 1:** \( u_i \) depends on the bandwidth allocated to user \( i \) and the network QoS.

In data communication networks, the user's utility not only depends on the allocated bandwidth but also depends on the network QoS. For example, a 2 Mbps bandwidth with the average delay 50 ms has better utility for most users than the same bandwidth with 500 ms average delay.

**Assumption 2:** \( u_i \) is a monotonic function of its variables.

This assumption is also quite intuitive. For instance, one would expect the utility to be monotonically increasing with bandwidth and monotonically decreasing with the QoS parameter, the average delay. In general, we do not assume strict monotonicity, since there may exist a point beyond which further increase in the QoS or bandwidth does not yield any additional benefit for the user. An example of this is the case of a constant bit rate application like VoIP – availability of bandwidth greater than that.
constant rate typically does not result in any improved performance.

Assumption 3: $u_i$ is concave function of its variables.

This assumption arises from a diminishing returns argument. We expect a user’s marginal utility to decrease with the bandwidth and QoS. It means that the more the bandwidth and better the quality, the less the user is willing to pay for further improvement. This assumption is also consistent with [37].

The well-known utility function in data communication networks, proposed by Kelly et al. [6], has the form $u_i = w_i \log x_i$, where $w_i$ is user’s willingness to pay and $x_i$ is the allocated bandwidth. Although this utility function fits into our assumptions about utility assumptions 3.2 and 3.3, it does not take QoS as a parameter into consideration for the utility function. In this paper, we redefine user’s utility as a function of the allocated bandwidth and the QoS of the network as follows

$$u = \frac{\beta}{T_{\text{now}}} \log \left( \frac{x}{\bar{x}} \right)$$

(3)

where $\beta > 0$ is the weighting factor and it describes the flow’s relative sensitivity to bandwidth and the QoS parameter. We use the present average delay $T_{\text{now}}$ to represent the QoS parameter of network. $\bar{x}$ is the minimum bandwidth requirement. Here we use the present average delay since users always keep record of the present network situation like round trip time, packet loss rate, etc.

Then user $i$’s utility in a multi-class network is the sum of utilities from each class of service as follows

$$u_i(x_i, \bar{x}_i) = \sum_{j=1}^{n} \beta_{ij} \frac{\log (x_{ij})}{T_{\text{now}}}$$

(4)

where $\beta_{ij}$ is a weighting factor and shows user $i$’s relative sensitivity to bandwidth and QoS in service class $j$; $\bar{x}_{ij}$ is user $i$’s minimum bandwidth requirement for class $j$ services. It is assumed that the network has an overall resource that exceeds the sum of the minimum bandwidth requirements.

As we stated before, the network supports $n$ classes of service. Within each class $j$, we assume that the traffic follows the Poisson distribution and this Poissonian arrival discipline is generally considered to be a good model for the aggregate traffic from a large number of independent users [38]. For simplicity, we further assume that the length of all packets in the network is exponentially distributed with average length equal to $1/\mu$. The network is modelled as a queuing network with First in First out (FIFO) discipline. Therefore the actual QoS parameter delay for each class $j$, $T_{\text{now}}$, is calculated using M/M/1 queuing model as follows

$$T_{\text{now}} = \frac{1}{\mu c_j - \sum_{k=1,k\neq j}^{m} x_{kj}}$$

(5)

where $c_j$ is the allocated resource for class $j$.

Together with (4) and (5), we can derive a more explicit description of each user $i$’s utility as follows

$$u_i(x_i, \bar{x}_i) = \sum_{j=1}^{n} \beta_{ij} \left( \frac{\mu c_j - \sum_{k=1,k\neq i}^{m} x_{kj}}{\bar{x}_{ij}} \right) \log \left( \frac{x_{ij}}{\bar{x}_{ij}} \right)$$

(6)

In order to keep different QoS for each class, we have the maximum possible arrival rate $s_j$ for each class as follows

$$T_{\text{SLA}} = \frac{1}{\mu c_j - s_j}$$

(7)

where $T_{\text{SLA}}$ is the QoS agreed to by service provider and users in Service Level Agreement for class $j$. Vector $s = (s_1, \ldots, s_n)$ represents maximum possible arrival rate for each class in the network. For simplicity, in the following analysis, we further assume the average message lengths are equal to 1 ($1/\mu = 1$). Therefore numerically, the message arrival rate is the same as bandwidth.

The service provider’s individual utility maximisation problem is represented as

$$\text{maximise } u(s, p) = s p^T$$

subject to the constraint

$$s \in S$$

(9)

where $s$ is a feasible set of available bandwidth supply in each class.

A competitive market equilibrium is a point that we denoted as $(p, s, x_1, \ldots, x_n)$ where $s = (s_1, \ldots, s_n)$ and $s_j$ is the total amount of bandwidth available for class $j$, and $p = (p_1, \ldots, p_n) \in R^n_+$ and $p_j$ is the unit price for class $j$ set by the market; such that:

1. (User optimality) $x_i$ is a maximiser of (4) given $\bar{x}_i$ and $p$ for every $i$.
2. (Service provider optimality) $s$ is a maximiser of (8) given $p$.
3. (Market efficiency) $p \geq 0, \sum_{i=1}^{m} x_{ij} \leq s_j, p_j (\sum_{i=1}^{m} x_{ij} - s_j) = 0$ for all $j$.

The last condition implies that for class $j$, when the supply is larger than the demand, the equilibrium price for class $j$ is equal to 0.
4 Equilibrium characteristics

In Section 3, we have given out assumptions about user’s utility in data communication networks and our proposed utility function in a multi-class network. In this section, we will investigate the existence and uniqueness of the competitive equilibrium of the multi-class network resource market.

**Theorem 1:** Using the utility functions defined in (6) and (8), the multi-class network resource market has a competitive equilibrium.

**Proof:** From the proof given in [29], if the utility function $u_i(x_i, \bar{x}_i)$ of each agent is continuous and concave in $x_i \in \mathbb{R}^n_+$ for every $\bar{x}_i \in \mathbb{R}^{(m-1)n}$, and each agent’s strategy set $X_i \subset \mathbb{R}^{an}$ is a closed and convex set. Since the network has fixed limited network resource, the supply for each class of service is also closed and convex set.

First, we check the monotonicity of the utility function $u_i(x_i, \bar{x}_i)$. The partial derivative of $u_i(x_i, \bar{x}_i)$ is as follows

\[
(\nabla x_i u_i(x_i, \bar{x}_i))_j = \beta_{ij} \left( \mu_j - \sum_{k=1, k \neq i}^{m} x_{kj} \right) > 0 \quad (10)
\]

Then, we check the concavity of the utility function $u_i(x_i, \bar{x}_i)$ using the second partial derivative of $u_i(x_i, \bar{x}_i)$, we have

\[
(\nabla^2 u_i(x_i, \bar{x}_i))_j = -\beta_{ij} \frac{x_{ij}}{x_j} \left( \mu_j - \sum_{k=1, k \neq i}^{m} x_{kj} \right) < 0 \quad (11)
\]

From (10) and (11), we find that each user’s utility function is monotonically increasing and is concave with respect to each variable $x_{ij}$, given $\bar{x}_i$ based on definition 1.

Now, we will check the monotonicity and concavity of service provider’s utility function $u_i$ as described in (8) as follows

\[
\frac{\partial u_i}{\partial x_j} = p_j \quad (12)
\]

\[
\frac{\partial^2 u_i}{\partial x_j^2} = 0 \quad (13)
\]

From (12) and (13), we also find that the service provider’s utility function is also monotonically increasing and concave with respect to each variable $s_j$ (although $u_i$ is not strictly concave to its variable $s_j$).

Note that $x_i$ is bounded under the linear constraint (2). It is a closed and convex set. Since the network has fixed limited network resource, the supply for each class of service $s$ is also closed and convex set.

Until now, we have proved that all agents’ (users and service provider) utility functions are continuous and concave to its variables (either $x_i$ or $s_j$). $x_i \in \mathbb{R}^n_+$ and $s \in \mathbb{R}^m_+$ are both closed and convex sets. We claim that this competitive multi-class network resource market has an equilibrium.

Note that if $p$ and $s$ are fixed and the users are the only agents in the game, the equilibrium problem reduces to a Nash equilibrium problem. By allowing $p$ and $s$ to change in the game, we can potentially achieve a more efficient equilibrium point. And each competitive equilibrium is Pareto optimal.

Now considering the optimality conditions of (6) and (8), we can find the following conditions using the Lagrangian multiplier. In other words

\[
\max_{x_i} u_i(x_i, \bar{x}_i) - \lambda (x_i^T p - w_i) + \gamma x_i^T \quad (14)
\]

where $\lambda \geq 0$ and $\gamma \geq 0$ are Lagrangian multipliers. Note that $\lambda$ is a scalar and $\gamma$ is a vector.

From the Karush–Kuhn–Tucker (KKT) [39] condition, we have

\[
\nabla x_i u_i(x_i, \bar{x}_i) - \lambda p^T + \gamma = 0 \quad (15)
\]

\[
\lambda (x_i p^T - w_i) = 0 \quad (16)
\]

\[
\gamma x_i^T = 0 \quad (17)
\]

Since the Lagrangian multiplier $\gamma \geq 0$, take it into (15), we have

\[
\nabla x_i u_i(x_i, \bar{x}_i) \leq \lambda p^T \quad (18)
\]

From (16) and (18), we obtain the following inequality

\[
(\nabla x_i u_i(x_i, \bar{x}_i)^T x_i) p \geq w_i \nabla x_i u_i(x_i, \bar{x}_i) \quad (19)
\]

Together with the constraints in (6) and (8), we have the complete necessary and sufficient conditions for a competitive equilibrium as follows

\[
(\nabla x_i u_i(x_i, \bar{x}_i)^T x_i) p \geq w_i \nabla x_i u_i(x_i, \bar{x}_i)
\]

\[
x_i p^T \leq w_i \quad (20)
\]

\[
x_i, p, s_j \geq 0, \forall i, j
\]

Now, multiplying $x_i \geq 0$ to both sides of (19), we have
\[ x_i^T p \geq \omega_i \] for all \( i \), and together with (20), we have
\[
\sum_{i=1}^{m} \omega_i \geq \sum_{j=1}^{n} x_j p_j \geq \sum_{j=1}^{n} \sum_{i=1,j \neq i}^{m} x_i p_j
\]
\[
= \sum_{i=1}^{m} x_i p_i \geq \sum_{i=1}^{m} \omega_i
\]

This means that every inequality in this sequence must be equal. Thus, we have the following characterisation of a competitive equilibrium.

**Theorem 2:** Every competitive equilibrium in multi-class network resource market has the following properties:

1. (Supply is equal to demand), \( \sum_{i} x_{ij} = s_j, \forall j \).
2. (All user’s budgets go to provider), \( sp^T = \sum_{i} \omega_i \).
3. (Every user only purchases the most valuable class resource), if \( x_{ij} > 0 \), then
\[
(\nabla_{x_i} u_i(x_i, \tilde{x}_i))^T p_j - \omega_i(\nabla_{x_i} u_i(x_i, \tilde{x}_i)) = 0.
\]

**Proof:** We have already showed the properties 1 and 2 above. We will only prove property 3 here.

From the KKT condition (17), if \( x_{ij} > 0 \), we have, \( \gamma = 0 \).

Now take this \( \gamma = 0 \) into (15), we have, \( \nabla_{x_i} u_i(x_i, \tilde{x}_i) - \lambda p^T = 0 \).

Together with (16), we obtain the property 3 as follows:
\[
(\nabla_{x_i} u_i(x_i, \tilde{x}_i))^T p_j - \omega_i(\nabla_{x_i} u_i(x_i, \tilde{x}_i)) = 0.
\]

We notice the necessary and sufficient equilibrium conditions (20) are all linear, except (19), i.e.,
\[
(\nabla_{x_i} u_i(x_i, \tilde{x}_i))^T p \geq \omega_i(\nabla_{x_i} u_i(x_i, \tilde{x}_i))
\]

We further assume that the multi-class network has fixed bandwidth supply for each class, that is \( S = \{ s \} \) is unique. We can now prove the uniqueness of the competitive equilibrium as follows.

We already have the partial derivative of \( u_i(x_i, \tilde{x}_i) \) from (10) as
\[
(\nabla_{x_i} u_i(x_i, \tilde{x}_i))_j = \frac{\beta_j}{x_{ij}} \left( \mu c_j - \sum_{k=1,k \neq i}^{m} x_{kj} \right), \forall j
\]

Multiply \( x_i \) to both sides of above equation, we find
\[
\nabla_{x_i} u_i(x_i, \tilde{x}_i)^T x_i = \sum_{j=1}^{n} \beta_j \left( \mu c_j - \sum_{k=1,k \neq i}^{m} x_{kj} \right)
\]

From the equilibrium property 1 of the Theorem 2, we have, \( \sum_{i=1}^{m} x_{ij} = s_j, \forall j \). Substituting it into (10) and (21), we obtain the following equations
\[
(\nabla_{x_i} u_i(x_i, \tilde{x}_i))_j = \frac{\beta_j}{x_{ij}} (\mu c_j - s_j + x_{ij}), \forall j
\]
\[
\nabla_{x_i} u_i(x_i, \tilde{x}_i)^T x_i = \sum_{j=1}^{n} \beta_j (\mu c_j - s_j + x_{ij})
\]

Now, we take the above two equations into (19) and write this non-linear inequality using the logarithmic transformation as
\[
\log(\omega_i) \leq \log(p_j) + \log\left( \frac{x_{ij}}{\beta_j (\mu c_j - s_j + x_{ij})} \right)
\]
\[
+ \log\left( \sum_{i=1}^{m} \beta_i (\mu c_j - s_j + x_{ij}) \right)
\]

We have already assumed that \( s \) is unique and, in Section 3, we defined the relationship between \( s_j \) and \( c_j \) in (7), therefore, the vector \( c \) is also fixed. We also find that the function on the right-hand side of (22) is a strictly concave function in \( x_{ij} \) and \( p_j \). The left-hand side of (22) is a constant, therefore (22) is a convex inequality. Based on the property of convex inequality in [40], we can now state the following theorem.

**Theorem 3:** In a multi-class network, with a fixed network resource for each class, the competitive equilibrium set is convex and the equilibrium can be computed in polynomial time.

To show the uniqueness of this solution, we proceed as follows.

From property 3 of Theorem 2, when \( x_{ij} > 0 \), we have
\[
\log(\omega_i) = \log(p_j) + \log\left( \frac{x_{ij}}{\beta_j (\mu c_j - s_j + x_{ij})} \right)
\]
\[
+ \log\left( \sum_{i=1}^{m} \beta_i (\mu c_j - s_j + x_{ij}) \right)
\]

Let \( [x^1, p^1] \) and \( [x^2, p^2] \) be two distinct competitive equilibriums. Since the equilibrium set is convex, the point \( [0.5x^1 + 0.5x^2, 0.5p^1 + 0.5p^2] \) is also an equilibrium, so that
\[
\log(\omega_i) = \log(0.5p^1_j + 0.5p^2_j)
\]
\[
+ \log\left( \frac{0.5x^1_{ij} + 0.5x^2_{ij}}{\beta_j (0.5c^1_j + 0.5c^2_j)} \right)
\]
\[
+ \log\left( \sum_{i=1}^{n} \beta_j (0.5s^1_j + 0.5s^2_j + 0.5x^1_{ij} + 0.5x^2_{ij}) \right)
\]

\( \forall x_{ij}, x_{ji}^2 \), satisfy either \( x_{ij}^1 > 0 \) or \( x_{ij}^2 > 0 \).
As showed before, the right-hand side of above equation is strictly concave in \( p \) and \( x \). From definition 2.2, we have (see equation at the bottom of the page)

\[
\forall x_{ij}^1, x_{ij}^2, \text{ satisfy either } x_{ij}^1 > 0 \text{ or } x_{ij}^2 > 0.
\]

Thus, we must have \( p^1 = p^2 \), and \( x^1 = x^2 \), which imply that the equilibrium point is unique. We can now state

**Theorem 4:** In a multi-class network, with fixed network resource for each class, the competitive price equilibrium \( p = (p_1, \ldots, p_m) \) and resource allocation \( x = (x_1, \ldots, x_m) \) is unique.

### 5 Budget allocation in competitive network resource market

In data communication networks, the bandwidth constraint exists either because of the limitation of the user equipment or the access speed of the network. In almost all cases, either the speed of the access network or the speed limitation of the user’s equipment is predefined. Therefore the bandwidth constraint for each user is fixed. In this section, we consider how to adjust the initial budget to satisfy each user’s bandwidth constraint.

Assume there is a budget agent who adjusts budget for each user to satisfy their bandwidth constraint \( b_i \) and we assume that \( \sum_i b_i \geq \sum_j s_j \). That means that the total users’ bandwidth constraint is higher or equal to the total available network bandwidth supply. A competitive market equilibrium \([w, p, x_1, \ldots, x_m]\) must satisfy:

1. (User optimality) \( x_i \) is a maximiser of (4) given \( \bar{x}_i, w \) and \( p \) for every \( i \).
2. (Service provider optimality) \( s \) is a maximiser of (8) given \( p \).
3. (Market efficiency) \( p \geq 0, \sum_{i=1}^{m} x_{ij} \leq s_j, p_j(\sum_{i=1}^{m} x_{ij} - s_j) = 0 \) for all \( j \).
4. (Budget adjustment) given \( x, w \) is a minimiser of

\[
\begin{align*}
\text{minimise } & w_i \sum \left( \max \left( 0, \sum j x_{ij} - b_i \right) \right) w_i; \\
\text{s.t. } & \sum w_i = m, w \geq 0
\end{align*}
\]

Equation (23) says that if user \( i \)'s access bandwidth constraint is broken, that is \( \sum_j x_{ij} - b_i \geq 0 \), then the budget agent will allocate less budget to user \( i \). And any budget allocation is optimal if \( \sum_j x_{ij} \leq b_i \) for all \( i \). That means every user’s access bandwidth constraint is met.

Since the budget adjustment problem is a bounded linear optimisation and all other maximisations are identical as described in Section 4, we can state the following theorem.

**Theorem 5:** The multi-class network resource market has a competitive equilibrium which satisfies access bandwidth constraint and each competitive equilibrium has the following properties.

1. (Supply is equal to demand), \( \sum_i x_{ij} = s_j, \forall j \).
2. (All users’ budgets go to provider), \( sp^T = \sum_i w_i \).
3. (Each user’s bandwidth constraint is met), \( \sum_j x_{ij} \leq b_i, \forall i \).
4. (Every user only purchases the most valuable class resource), if \( x_{ij} > 0, \) then

\[
(V_u u_i(x_i, \bar{x})^Tx_i)p_j - w_i(V_u u_i(x_i, \bar{x}))j = 0.
\]

As already shown in Section 4 that in a multi-class network, with a fixed network resource for each class, the competitive price equilibrium \([p, x_1, \ldots, x_m]\) is unique and can be calculated in polynomial time. We use the following iterative algorithm for budget allocation to satisfy the bandwidth constraint:

\[
\begin{align*}
\log(0.5p^1 + 0.5p^2) + & \log \left( \frac{0.5x_{ij}^1 + 0.5x_{ij}^2}{\beta_j(\mu c_j - s_j + 0.5x_{ij}^1 + 0.5x_{ij}^2)} \right) + \log \left( \sum \beta_j(\mu c_j - s_j + 0.5x_{ij}^1 + 0.5x_{ij}^2) \right) \\
> & 0.5(\log(p^1) + \log \left( \frac{x_{ij}^1}{\beta_j(\mu c_j - s_j + x_{ij}^1)} \right) + \log \left( \sum \beta_j(\mu c_j - s_j + x_{ij}^1) \right)) + 0.5(\log(p^2)) \\
+ & \log \left( \frac{x_{ij}^2}{\beta_j(\mu c_j - s_j + x_{ij}^2)} \right) + \log \left( \sum \beta_j(\mu c_j - s_j + x_{ij}^2) \right) \geq 0.5 \log(w_i) + 0.5 \log(w_i) = \log(w_i)
\end{align*}
\]
Algorithm 1: Initialise budget assigned to each user \( w_i = 1, \) \( i = 1, \ldots, m. \)

repeat

Compute competitive economy equilibrium \([x_1, \ldots, x_m, p]\) under \( s = (s_1, \ldots, s_n) \) and \( w = (w_1, \ldots, w_m). \)

Obtain the allocated bandwidth to each user \( i, \sum_j x_{ij}; \)

Calculate average bandwidth surplus, \( \text{avg}_i = \frac{\sum_j (x_{ij} - s_i)}{m}; \)

Update \( w_i = w_i + \frac{\sum_j x_{ij} - \text{avg}_i}{m}, \) \( i = 1, \ldots, m. \)

Until \( b_i - \sum_j x_{ij} \geq \text{error tolerance}, i = 1, \ldots, m. \)

In each iteration, the unique competitive equilibrium is derived given network resource for each class \( s_j \) and budget for each user \( w_i. \) The user budget is reassigned according to the bandwidth shortage of each user in the equilibrium solution. The idea of comparing the user’s bandwidth surplus with average surplus means less budget allocation to the users with lower bandwidth surplus whereas keeping the total budget unchanged. Here, \( k \) is a scalar parameter. We can adjust \( k \) to obtain the competitive equilibrium with bandwidth constraint with different pace.

6 Illustrative example

This section presents a numerical example about the pricing and resource allocation in a multi-class network resource market as described in this paper. We will show the competitive equilibrium properties of this market.

We assume the network supports three different service classes \((n = 3)\) with the capacity \( c = 100. \) The service class 1 supports real-time gaming and the QoS parameter average delay defined in SLA as \( T_{1SLA} = 0.04 s; \) the service class 2 and service class 3 are designed to carry interactive streaming service and non-interactive streaming service respectively with \( T_{2SLA} = 0.1 \) and \( T_{3SLA} = 0.2. \) To emphasise the method, we assume there are three users \((m = 3)\) competing for the network resource. Based on the utility function proposed in (4), the utility functions for the users are

\[
\begin{align*}
  u_1 &= (\mu c_1 - x_{11} - x_{31}) \log\left(\frac{x_{11}}{4}\right) \\
  &+ (\mu c_2 - x_{12} - x_{32}) \log\left(\frac{x_{12}}{2}\right) \\
  &+ (\mu c_3 - x_{13} - x_{33}) \log\left(\frac{x_{13}}{5}\right) \\
  u_2 &= 1.1(\mu c_1 - x_{21} - x_{31}) \log\left(\frac{x_{21}}{4}\right) \\
  &+ (\mu c_2 - x_{22} - x_{32}) \log\left(\frac{x_{22}}{2}\right) \\
  &+ (\mu c_3 - x_{23} - x_{33}) \log\left(\frac{x_{23}}{5}\right) \\
  u_3 &= 1.2(\mu c_1 - x_{31} - x_{21}) \log\left(\frac{x_{31}}{4}\right) \\
  &+ (\mu c_2 - x_{32} - x_{22}) \log\left(\frac{x_{32}}{2}\right) \\
  &+ (\mu c_3 - x_{33} - x_{23}) \log\left(\frac{x_{33}}{5}\right)
\end{align*}
\]

As shown in (24)–(26), all three users have the minimum bandwidth requirement for class 1 service as \( x_{11} = x_{31} = 31 = 4, \) class 2 as \( x_{12} = x_{32} = x_{22} = 2 \) and class 3 as \( x_{13} = x_{33} = x_{23} = 5. \) The weighting factors of the first user for different class service as \( \beta_{11} = \beta_{12} = \beta_{13} = 1. \) The second user and the third user have higher weighting factors for class one service and identical weighting factors for class 2 and 3 services as, \( \beta_{21} = 1.1, \beta_{31} = 1.2, \) and \( \beta_{22} = \beta_{23} = \beta_{32} = \beta_{33} = 1. \)

As proven in Section 4, when the network resource allocation among classes is fixed, the market has an unique competitive equilibrium. We further assume the network allocates 45% of network resource to class 1 service, 30% to class 2 service and 25% to class 3 service. To satisfy the QoS agreement in each class’s SLA, based on (7), the available network resource for each class is 20 as shown in Table 1. We also assume the average length of packets in the network \( 1/\mu = 1 \) and the initial endowments for three users are \( w_1 = 8, w_2 = 10 \) and \( w_3 = 12. \) Then the competitive solution is as follows:

\[
\begin{align*}
  p_1 &= 0.8294253, \quad p_2 = 0.3939162, \quad p_3 = 0.2766585 \\
  x_{11} &= 5.1842, \quad x_{12} = 5.6653, \quad x_{13} = 5.3071 \\
  x_{21} &= 6.6393, \quad x_{22} = 6.7135, \quad x_{23} = 6.6822 \\
  x_{31} &= 8.1765, \quad x_{32} = 7.6207, \quad x_{33} = 8.0107
\end{align*}
\]

The above results show that under competitive equilibrium:

1. Each user spent all the budget: \( w_i = \sum_j p_j x_{ij}, \) \( \forall i, 1, 2, 3. \)

2. All users budget goes to service provider: \( w_i = \sum_j p_j x_{ij}. \)

3. Demand is equal to supply for each class bandwidth: \( s_j = \sum_i x_{ij}, \forall j, 1, 2, 5. \)

4. Each class’s QoS is well-maintained: \( D_j = 1/((\mu c_j - \sum_i x_{ij}) = T_{jSLA}, \forall j, 1, 2, 3. \)

Table 1 Resource allocation among classes

<table>
<thead>
<tr>
<th></th>
<th>Class 1</th>
<th>Class 2</th>
<th>Class 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{1SLA} )</td>
<td>0.04 s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( T_{2SLA} )</td>
<td></td>
<td>0.1 s</td>
<td></td>
</tr>
<tr>
<td>( T_{3SLA} )</td>
<td></td>
<td></td>
<td>0.2 s</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_2 )</td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_3 )</td>
<td>25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s_1 )</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s_2 )</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s_3 )</td>
<td>20</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
From the solution we also obtain the total bandwidth for the first user is $\sum_{j=1}^{3} x_{1j} = 16.1571$, the second user is $\sum_{j=1}^{3} x_{2j} = 20.0349$ and the third user is $\sum_{j=1}^{3} x_{3j} = 23.8080$. We can also calculate the utility for the first user as $u_1 = 35.7132$, for the second user as $u_2 = 59.5291$ and for the third user as $u_3 = 83.9198$; and therefore the social utility has the value $u_1 + u_2 + u_3 = 179.6212$.

Now consider each of them has a bandwidth constraint $b_1 = 15$, $b_2 = 20$ and $b_3 = 30$. From the above example, if the budget agent sets $w_1 = 8$, $w_2 = 10$ and $w_3 = 12$, the bandwidth allocation cannot satisfy the bandwidth constraint. By the algorithm proposed in Section 5, we can adjust the initial budget endowments and obtain the competitive results as follows. In this example, we set the error tolerance as 0.01

$$w_1 = 7.474, \quad w_2 = 9.984, \quad w_3 = 12.542$$

$$\rho_1 = 0.829419, \quad \rho_2 = 0.393402, \quad \rho_3 = 0.277179$$

$$x_{11} = 4.8997, \quad x_{12} = 5.2783, \quad x_{13} = 4.8115$$

$$x_{21} = 6.6319, \quad x_{22} = 6.7182, \quad x_{23} = 6.6398$$

$$x_{31} = 8.4684, \quad x_{32} = 8.0035, \quad x_{33} = 8.5487$$

As shown above, when we adjust the initial budget endowments to $w_1 = 7.474$, $w_2 = 9.984$ and $w_3 = 12.542$, on the competitive equilibrium, the bandwidth allocation for all three users allocated bandwidth satisfy their bandwidth constraints: $\sum_{j=1}^{3} x_{1j} = 14.9894 \leq b_1 = 15$, $\sum_{j=1}^{3} x_{2j} = 19.99 \leq b_2 = 20$ and $\sum_{j=1}^{3} x_{3j} = 25.0206 \leq b_3 = 30$.

### 7 Conclusion

This paper has considered multi-class network resource in a competitive market where each user endowed with an initial budget will purchase bandwidth from each class of the network resource to maximise its utility function. After defining utility functions for users in a multi-class network, we have proved that there exists a unique competitive equilibrium $[\rho, x_1, \ldots, x_n]$ when we fix the available bandwidth for each class. This competitive equilibrium can be calculated in polynomial time. For bandwidth constraint because of user’s equipment or the speed of the access network, the proposed algorithm adjusts the initial budget for users to satisfy their respective constraints. Further, we have proved that a competitive equilibrium $[w, \rho, x_1, \ldots, x_n]$, under bandwidth constraints, always exists for the multi-class resource network as long as the sum of the total bandwidth constraints is equal to or exceeds the total bandwidth supply. Since the competitive equilibrium is Pareto optimal, the proposed solution achieves both higher social utility and better individual satisfaction than the Nash equilibrium.

### 8 Acknowledgment

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